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Wave-induced vorticity in free-surface boundary layers: application to mass transport in edge waves

By B. D. DORE
Department of Mathematics, University of Reading, Berkshire, England

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The time-averaged vorticity field within the free-surface boundary layer associated with a general class of propagating gravity waves is considered. The principal results are applied in a calculation of the mass transport velocity field for edge waves.

1. Introduction

The generation of a secondary time-averaged vorticity field within laminar boundary layers adjacent to oscillating free surfaces has been investigated by Longuet-Higgins (1953) in circumstances when the fluid motion is two-dimensional. For such motions, the value of this vorticity at the edge of the oscillatory surface layer is, in general, non-zero. Consequently, with reference to the time-averaged motion in the interior of the fluid, Longuet-Higgins (1960) noted that the presence of this boundary layer is equivalent to a mean tangential stress. Phillips (1966) and Longuet-Higgins (1969) made use of this equivalence in applications to progressive gravity waves in deep water.

In the present note, the generation of mean vorticity within the free-surface boundary layer is considered for a much wider class of gravity waves, comprising those which propagate in the sense described in § 2. The calculation forms a generalization of that of Dore (1974), who considered two interacting wave trains propagating in arbitrary directions. In a limiting case, the present results reduce to those of Longuet-Higgins (1953) for arbitrary two-dimensional fluid motions beneath an oscillating free surface. By way of application, we determine a solution for the mass transport velocity field associated with the propagation of a Stokes edge wave over a plane beach of arbitrary slope. Also, an approximate solution is given for edge-wave modes of higher order propagating over beaches of very small slope.

2. Formulation

We consider wave motion in a homogeneous incompressible fluid of finite, but not necessarily uniform, depth. The motion is first referred to a stationary system of Cartesian co-ordinates (x', y', z') having origin at the (local) mean level of the free surface and z' axis directed vertically upwards.

Neglecting friction for the moment, we consider the fluid motion induced by some propagating wave (of period $2\pi/\sigma$) whose amplitude and phase functions are independent of and linear in x', respectively. Henceforth, physical variables will be non-dimensionalized with respect to σ and k, where $2\pi/k$ denotes the wavelength in the x direction. The motion, assumed irrotational, is described in terms of a velocity potential Φ satisfying Laplace's equation $\nabla^2\Phi = 0$. Further, Φ is expanded in terms of an ordering parameter α, a measure of the maximum wave slope, as

$$\Phi = \sum_{n=1}^{\infty} \alpha^n \Phi_n(x, y, z, t),$$

similar to the expansion used by Stokes (1847). Then, for a free-surface displacement

$$z_s = \alpha f(y)\, e^{i(x-t)} + O(\alpha^2),$$

where $f(y)$ may be complex valued, we have

$$\Phi_1 = \phi_1(y, z)\, e^{i(x-t)},$$

$$-if(y) = \partial\phi_1/\partial z = \sigma^2\phi_1/gk \quad (z = 0).$$

(Wherever complex terms represent physical variables it is understood that only the real parts are to be taken.) To $O(\alpha^2)$, the mass transport velocity is given by

$$\alpha^2\mathbf{Q}_l = \langle\mathbf{q}\rangle + \langle(\textstyle\int \mathbf{q}\, dt\,.\,\nabla)\,\mathbf{q}\rangle,$$

where $\langle\mathbf{q}\rangle$ denotes the time-averaged value of the velocity vector $\mathbf{q}$ at a point fixed in space. The second term on the right-hand side is denoted by $\alpha^2\mathbf{Q}_s$ and is referred to as the Stokes drift velocity; thus

$$\mathbf{Q}_s = (U_s, V_s, W_s) = \tfrac{1}{2}(\nabla\Phi_{1x}\,.\,\nabla)\,\nabla\Phi_1^*,$$

where the asterisk denotes the complex conjugate.

3. The free-surface boundary layer

It is assumed that the quantity $\epsilon^{-2} = \sigma/2\nu k^2$, in which ν denotes the kinematic viscosity, is a representative wave Reynolds number and is $\gg 1$. Consequently, there is an oscillatory free-surface boundary layer of thickness $O(\epsilon)$. For purposes of this section, it can be shown to be immaterial whether the wave motion is (*a*) periodic in time, with a small spatial damping rate $O(\epsilon)$, (*b*) completely maintained by the action of certain wind forces, which give rise to suitable surface stresses,† or (*c*) a free oscillation, decaying on the time scale $O(\epsilon^{-1})$. However, with a view to applications to steady-state calculations such as that made in §4, the free oscillation (*c*) must be disregarded, since the decay time is much shorter than the time scale (typically $O(\epsilon^{-2})$) needed for the establishment of a steady rotational velocity field $O(\alpha^2)$ throughout the entire fluid. For definiteness, we shall refer to (*b*) in the following description.

† These may consist of normal stresses alone, or of normal and tangential stresses. But, in the latter case, the class of tangential stresses would be chosen so that the results of equation (3.4) are unaltered.

Longuet-Higgins (1953) showed that, owing to the fluctuating position of the free surface and to the existence of the free-surface boundary layer, it is necessary to use a special co-ordinate system in which the free surface is a co-ordinate surface if the wave amplitude is comparable with, or greatly exceeds, the boundary-layer thickness. In order that the present analysis of the layer should be valid for such amplitudes, we shall also introduce a suitable co-ordinate system. Thus the motion is first considered with respect to a frame of reference moving with unit velocity in the $+x$ direction, and is steady relative to this frame. The equation for the vorticity $\boldsymbol{\omega} = \text{curl}\,\mathbf{q}$ becomes

$$\text{curl}\,(\mathbf{q}\times\boldsymbol{\omega}) = \tfrac{1}{2}\epsilon^2\,\text{curl}\,\text{curl}\,\boldsymbol{\omega}. \tag{3.1}$$

Within the free-surface layer, the dynamical equations are then considered in terms of a (non-orthogonal) curvilinear co-ordinate system (ξ, η, ζ), which is of a more general form than that employed by Dore (1974). We write

$$\xi = X - \sum_{n=1}^{2} \alpha^n \Phi_n(X, y, z),$$
$$\eta = y + i\alpha\,\partial\Phi_1/\partial y + O(\alpha^2),$$
$$\zeta = z - i\alpha\,\partial\Phi_1/\partial z + O(\alpha^2),$$

where $X = x - t$. The ξ surfaces are chosen here to coincide with the equi-potential surfaces of the (inviscid) wave motion. Terms up to $O(\alpha^2)$ in the expansions for η and ζ are chosen such that (i) normals to ζ surfaces are orthogonal to normals to both ξ and η surfaces and (ii) the position of the free surface is represented by $\zeta = 0$. Then, to $O(\alpha^2)$, ζ surfaces are stream surfaces of the inviscid motion.

A boundary-layer variable $Z = \zeta/\epsilon$ is now introduced, and the vorticity vector is expanded in the form

$$\boldsymbol{\omega} = \sum_{j,k} \alpha^j \epsilon^k \boldsymbol{\omega}_{jk}(\xi, \eta, Z) \quad (j = 1, 2, \ldots;\ k = 0, 1, \ldots).$$

Equation (3.1) is then considered to $O(\alpha^j)$, $j = 1, 2$, as described by Dore (1974). The vorticity vector is written as

$$\boldsymbol{\omega} = \omega_\xi \mathbf{i}_\xi + \omega_\eta \mathbf{i}_\eta + \omega_\zeta \mathbf{i}_\zeta,$$

where $\mathbf{i}_\xi$, etc., are unit vectors parallel to the co-ordinate curves, and averages with respect to ξ are denoted by $\overline{\omega}_\xi$, etc. The equations for $\omega_{\xi 20}$ and $\omega_{\eta 20}$, when so averaged, are readily integrated through the boundary layer, and give

$$\overline{\omega}_{\xi 20}\big|_0^{Z_\infty} = \overline{[(Jh_\xi h_\eta^2)_1]_{\zeta=0}\,[\omega_{\xi 10}]_{Z=0}}, \quad \overline{\omega}_{\eta 20}\big|_0^{Z_\infty} = \overline{[(Jh_\xi^2 h_\eta)_1]_{\zeta=0}\,[\omega_{\eta 10}]_{Z=0}}, \tag{3.2 a, b}$$

where $J = \partial(\xi, \eta, \zeta)/\partial(x, y, z)$ and h_ξ, etc., are the Jacobian and scale factors of the co-ordinate transformation, respectively, and $Z = Z_\infty$ represents the edge of the layer. Now the condition that the free surface is subject to no tangential stress requires that

$$\mathbf{i}_\xi \frac{h_\xi}{h_\zeta}\frac{\partial}{\partial\zeta}\left(\frac{q_\xi}{h_\xi}\right) + \mathbf{i}_\eta \frac{h_\eta}{h_\zeta}\frac{\partial}{\partial\zeta}\left(\frac{q_\eta}{h_\eta}\right) = 0 \quad (\zeta = 0). \tag{3.3}$$

This condition can be conveniently expressed by writing $\mathbf{q} = -\nabla\xi + \tilde{\mathbf{q}}$ (where $\tilde{\mathbf{q}}$ is rotational) and using the property that the shape of the free surface is given by the inviscid theory. After making use of expressions for the vorticity components in curvilinear co-ordinates, application of the condition yields

$$[\overline{\omega}_{\xi 20}]_{Z=0} = \overline{[(Jh_\xi h_\eta^2)_1]_{\zeta=0}[\omega_{\xi 10}]_{Z=0}} - \overline{[\partial(\nabla\xi\,.\,\nabla\eta)_2/\partial\zeta]_{\zeta=0}} - \overline{[(\mathbf{i}_\xi\,.\,\mathbf{i}_\eta)_1]_{\zeta=0}[\omega_{\eta 10}]_{Z=0}},$$

with an equation of similar form for $[\overline{\omega}_{\eta 20}]_{Z=0}$. The values of the left-hand sides can thus be found in terms of the co-ordinate transformation and the linear vorticity field $\boldsymbol{\omega}_{10}$. Then, from (3.2), the values

$$[\overline{\omega}_{\xi 20}]_{Z=Z_\infty} = \frac{1}{2}\left[\frac{\partial}{\partial y}(\nabla\Phi_{1x}\,.\,\nabla\Phi_{1z}^*) - \frac{\partial}{\partial z}(\nabla\Phi_{1x}\,.\,\nabla\Phi_{1y}^*)\right]_{z=0}, \tag{3.4a}$$

$$[\overline{\omega}_{\eta 20}]_{Z=Z_\infty} = \tfrac{1}{2}[\partial(\nabla\Phi_{1x}\,.\,\nabla\Phi_{1x}^*)/\partial z]_{z=0} \tag{3.4b}$$

are found at the edge of the layer, in terms of the linear velocity potential Φ_1 of inviscid theory. Equations (3.4) represent boundary conditions for the rotational second-order motion outside the oscillatory boundary layers at the surface and bottom.

Mean vorticity. The linear periodic vorticity field $\boldsymbol{\omega}_1$ vanishes at the edge of the free-surface boundary layer. We therefore deduce the time-averaged values

$$\langle\omega_x\rangle = \alpha^2[\partial W_s/\partial y - \partial V_s/\partial z]_{z=0}, \quad \langle\omega_y\rangle = \alpha^2[\partial U_s/\partial z]_{z=0} \tag{3.5a, b}$$

for the horizontal Cartesian components (correct to $O(\alpha^2)$) of the wave-induced vorticity field at the edge of the surface layer. (The corresponding vertical component $\langle\omega_z\rangle$ can be calculated after the solution for $\langle\mathbf{q}_2\rangle$ has been determined in the whole fluid.) These values are essential in any attempted calculation of the time-averaged motion of individual elements throughout the fluid. With reference to the mass transport velocity, we may obtain the relations

$$\left.\begin{aligned} \partial U_l/\partial z &= 2\partial U_s/\partial z \qquad &(3.6a)\\ \partial V_l/\partial z &= 2(\partial V_s/\partial z - \partial W_s/\partial y) \qquad &(3.6b)\end{aligned}\right\} \quad \text{on} \quad z = 0,$$

provided that spatial rates of change just beyond the oscillatory layer are $O(1)$ or, in other words, that no second boundary layer, adjacent to the oscillatory layer, exists. The first relation is a generalization of those obtained for long- and short-crested progressive waves by Longuet-Higgins (1953) and Dore (1974), respectively. The quantity

$$[\partial W_s/\partial y]_{z=0} = \left[\tfrac{1}{2}i\frac{\partial}{\partial y}\left(\frac{\partial\phi_1}{\partial z}\frac{\partial^2\phi_1^*}{\partial z^2}\right)\right]_{z=0}$$

vanishes in many applications (see, for example, §4 below), but is in general non-zero when surfaces of constant phase are not vertical, as occurs in the case of surface-wave propagation in water of variable depth (see, for example, Keller 1958).

When a suitable modification is made to the non-dimensional scheme, a calculation similar to that described above may be carried out and the limit $k \to 0$ taken. Equations identical with (3.5 *a*) and (3.6 *b*) hold for the resulting two-dimensional motion of *arbitrary* oscillatory form, and agree with the work of Longuet-Higgins (1953).

Wave-induced mean effects of oscillatory boundary layers on the interior fluid have been discussed by Longuet-Higgins (1953), who considered two-dimensional motion in fluid bounded by both free and fixed surfaces. The above formulae (3.4)–(3.6) are complementary to those calculated for fixed two-dimensional boundaries by Hunt & Johns (1963). At the edges of the layers, values $O(\alpha^2)$ of the tangential components of the mean velocity and mean vorticity are predicted in the cases of fixed boundaries and (clean) free surfaces, respectively. Such values represent boundary conditions imposed on the mean motion outside the oscillatory boundary layers. To determine this motion, it is generally necessary to require that $\alpha \ll \epsilon$ if complications arising, for example, from secondary boundary layers (adjacent to the oscillatory layers, and suggested by Stuart 1966) are to be avoided. With this restriction on wave amplitude, solutions for the mean motion may be sought in terms of the conduction equation, analogous to that of Longuet-Higgins (1953), in which the steady vorticity field $O(\alpha^2)$ has been established predominantly through the medium of viscous conduction. With reference to Cartesian co-ordinates, the vorticity equation then yields

$$\operatorname{curl}\operatorname{curl}\operatorname{curl}\langle \mathbf{q}_{20}\rangle = 0. \tag{3.7}$$

Before considering a specific example, it should, however, be pointed out that the mean motion in the interior fluid may, in a wider context, be only weakly determined by viscous conduction of vorticity. For instance, in a rotating fluid, this motion is dominated by Coriolis forces if the Ekman number is small, and the constraints imposed on the mass transport velocity field by even remote boundaries override viscous effects; moreover, in an inhomogeneous fluid, mean vertical motion of fluid elements is inhibited by the stratification.

4. Application to edge waves

As an application of the results of § 3, we first consider the case of a Stokes edge wave propagating in the $+x$ direction over a plane bottom $z = -y\tan\beta$ $(0 < \beta < \frac{1}{2}\pi)$. For this wave,

$$\left.\begin{aligned} f(y) &= \exp(-y\cos\beta), \\ \phi_1(y,z) &= (-i/\sin\beta)\exp[-(y\cos\beta - z\sin\beta)], \end{aligned}\right\} \tag{4.1}$$

$\sigma^2 = gk\sin\beta$ and we assume that $\alpha \ll \beta$. The Stokes drift velocity is everywhere in the direction of wave propagation and is given by

$$U_s = (\sin^2\beta)^{-1}\exp[-2(y\cos\beta - z\sin\beta)]. \tag{4.2}$$

Thus, according to inviscid theory, the time-averaged particle paths are straight and parallel to the x axis. The Stokes drift velocity has been calculated for more general progressive edge waves by Kenyon (1969), who assumed that $\beta \ll 1$ and used a hydrostatic approximation. His calculation was partially motivated by a suggestion of Ursell (1952) that nonlinear effects might be important for edge waves. Indeed, in a rather extreme oceanic situation, Kenyon found that the value of the long-shore drift velocity near the shoreline could be about 15 cm/s for Stokes' edge wave.

For standing edge waves in viscous fluids, the mass transport velocity in the bottom boundary layer has been considered by Bowen & Inman (1971), who proposed a possible explanation of certain rhythmical sedimentary features on sloping beaches. No calculations of the complete mass transport velocity field appear to exist for edge waves in viscous fluids, although this topic was also mentioned by Ursell (1952). We now consider some aspects of this problem which result from the diffusion of mean vorticity through the whole fluid, and seek the solution of (3.7) for a Stokes progressive edge wave.

Long-shore drift. It may readily be shown that $\langle u_{20}\rangle$ is zero throughout the bottom boundary layer. In terms of polar co-ordinates such that $y = R\cos\chi$ and $z = R\sin\chi$, the relevant boundary-value problem in the y, z plane is

$$\nabla_2^2\langle u_{20}\rangle = \text{constant} \quad (0 > \chi > -\beta),$$
$$\partial\langle u_{20}\rangle/\partial\chi = (2R/\sin\beta)\exp(-2R\cos\beta) \quad (\chi = 0),$$
$$\langle u_{20}\rangle = 0 \quad (\chi = -\beta),$$

where $\nabla_2^2 \equiv \partial^2/\partial R^2 + R^{-1}\partial/\partial R + R^{-2}\partial^2/\partial\chi^2$ and the inhomogeneous boundary condition arises on account of the mean vorticity $\langle\omega_{y20}\rangle$ at the edge of the surface layer. The solution which is finite at $R = 0$ and which tends to zero as $R \to \infty$ can be obtained, for example, by means of the Mellin transform. We find the alternative representations

$$\langle u_{20}\rangle = \frac{2\cos(\pi\chi/2\beta)}{\beta\sin 2\beta}\int_0^\infty \frac{s^{\pi/2\beta}(1-s^{\pi/\beta})\exp(-2Rs\cos\beta)}{1-2s^{\pi/\beta}\cos(\pi\chi/\beta)+s^{2\pi/\beta}}\frac{ds}{s}, \tag{4.3}$$

$$\langle u_{20}\rangle = \frac{2R}{\sin\beta}\left[\sum_{n=1}^{\infty}\left\{\frac{(-2R\cos\beta)^{n-1}\sin n(\beta+\chi)}{n!\cos n\beta}\right\} - \sum_{m=0}^{\infty}\{(2R\cos\beta)^{M-1}\Gamma(-M)\cos M\chi\}\right], \tag{4.4}$$

where $M = (2m+1)\pi/2\beta$. (If $2\beta/\pi$ is a rational number c/d, where c (odd) and d ($> c$) are positive integers with no common factor, the Mellin transform $\mathscr{M}(\langle u_{20}\rangle; s)$ has an infinite sequence of double poles at

$$s = -(2N-1)d \quad (N = 1, 2, \ldots),$$

and (4.4) must be modified accordingly.) The above representations yield

$$\langle u_{20}\rangle \sim \begin{cases} \dfrac{2\cos(\pi\chi/2\beta)\,\Gamma(\pi/2\beta)}{\beta\sin 2\beta\,(2R\cos\beta)^{\pi/2\beta}} & \text{as} \quad R\to\infty, \\ 4R\operatorname{cosec} 2\beta\sin(\beta+\chi) & \text{as} \quad R\to 0. \end{cases} \tag{4.5}$$

The mass transport velocity $U_l = \langle u_{20}\rangle + U_s$ is everywhere in the direction of wave propagation, and by (4.3) is greater than the corresponding value of U_s. It may be seen from (4.5) that U_l decays algebraically as $R\to\infty$, in contrast to the exponential decay predicted by inviscid theory. In figure 1, the relative values of the long-shore drift velocity U_l are presented for three locations for the case $\beta = \frac{1}{4}\pi$. The corresponding values given by inviscid theory lie between the lower two curves in the figure. For very small β, there is virtually no difference between the values of U_l and U_s associated with Stokes' edge wave (see §4.1 below). Within the bottom boundary layer,

$$U_l = U_s = \operatorname{cosec}^2\beta(1-2e^{-Z}\cos Z + e^{-2Z})\,e^{-2\hat{y}} \quad (Z = \hat{z}/\epsilon),$$

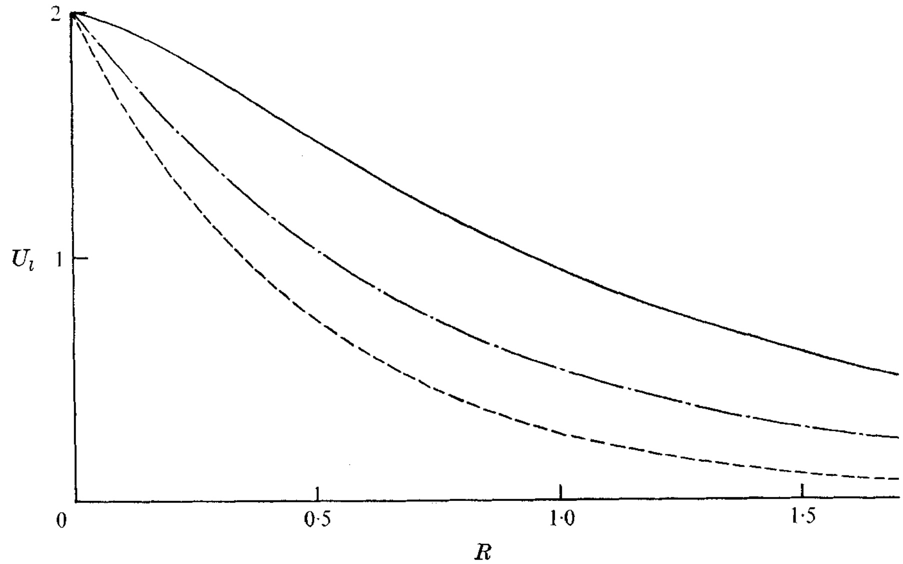

FIGURE 1. Values of U_l in a Stokes edge wave ($\beta = \frac{1}{4}\pi$) for fluid particles near the free surface (solid curve), at mid-depths (dot–dashed curve) and just above the bottom boundary layer (dashed curve).

and is everywhere in the direction of wave propagation. The co-ordinates $\hat{y}$ and $\hat{z}$ are measured along and perpendicular to the bottom $\hat{z} = 0$.

Mass transport in transverse planes x = constant. It is found that the Stokes drift velocity $\hat{V}_s$ is everywhere zero, and that

$$\hat{V}_l = \langle \hat{v}_{20} \rangle = 2\,\mathrm{cosec}^2\beta(\tfrac{1}{4} - e^{-Z}\sin Z - \tfrac{1}{4}e^{-2Z})\,e^{-2\hat{y}}$$

within the bottom boundary layer. Outside the layers, the boundary-value problem for the mass transport velocity can be formulated in terms of a stream function $\Psi(R, \chi)$ such that $\langle q_{R20} \rangle = R^{-1}\,\partial\Psi/\partial\chi$ and $\langle q_{\chi 20} \rangle = -\partial\Psi/\partial R$. Thus

$$\nabla_2^4 \Psi = 0 \quad (0 > \chi > -\beta),$$
$$\Psi = 0 \quad (\chi = 0,\ -\beta),$$
$$\partial^2\Psi/\partial\chi^2 = 0 \quad (\chi = 0),$$
$$\partial\Psi/\partial\chi = \tfrac{1}{2}\,\mathrm{cosec}^2\beta R\,e^{-2R} \quad (\chi = -\beta).$$

The solution which is finite at $R = 0$ and which tends to zero as $R \to \infty$ is readily obtained after application of the Mellin transform:

$$\Psi = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{R}{2\sin^2\beta}\,\frac{\sin(s-1)\chi\sin(s+1)\beta - \sin(s+1)\chi\sin(s-1)\beta}{s\sin 2\beta - \sin 2s\beta}\,\frac{\Gamma(s)}{(2R)^s}\,ds. \tag{4.6}$$

In the first quadrant, the denominator has zeros at $s = s_n$, where

$$2n\pi < 2\,\mathrm{Re}(s_n)\beta < (2n + \tfrac{1}{2})\pi \quad \text{and} \quad \mathrm{Im}(s_n) \neq 0 \quad (n = 1, 2, \ldots).$$

There may be a finite number of zeros, other than $s = 0, 1$, on the real axis, but this occurs only when β is very close to $\frac{1}{2}\pi$. Excluding $s = 1$, the zeros of the denominator with smallest positive real part occur at $s = s_1, s_1^*$. We must therefore take $0 < a < \mathrm{Re}\,(s_1)$, and find

$$\Psi = \frac{R}{\sin^2\beta}\Bigg[\frac{\chi\cos\chi\sin\beta-\beta\sin\chi\cos\beta}{\sin 2\beta-2\beta} - R\frac{\chi\sin 2\beta-\beta\sin 2\chi}{\sin 2\beta-2\beta\cos 2\beta}$$
$$+\frac{1}{2}\sum_{n=2}^{\infty}\frac{(-2R)^n}{n!}\frac{\sin(n+1)\chi\sin(n-1)\beta-\sin(n-1)\chi\sin(n+1)\beta}{\sin 2n\beta-n\sin 2\beta}$$
$$+\mathrm{Re}\left\{\sum_{n=1}^{\infty}\Gamma(-s_n)(2R)^{s_n}\frac{\sin(s_n+1)\chi\sin(s_n-1)\beta-\sin(s_n-1)\chi\sin(s_n+1)\beta}{\sin 2\beta-2\beta\cos 2s_n\beta}\right\}\Bigg].$$

The radial component of $\mathbf{Q}_l$ near the surface is therefore inwards for $R \ll 1$. For the distant field, asymptotic expansion of the Bromwich integral (4.6) yields

$$\Psi \sim \frac{R}{\sin^2\beta}\mathrm{Re}\left\{\frac{\sin(s_1-1)\chi\sin(s_1+1)\beta-\sin(s_1+1)\chi\sin(s_1-1)\beta}{2\beta\cos 2s_1\beta-\sin 2\beta}\frac{\Gamma(s_1)}{(2R)^{s_1}}\right\}$$

as $R\to\infty$, so that the algebraic decay is oscillatory. The velocity field of the mean transverse flow therefore has an infinite sequence of stagnation points on the time-averaged level of the free surface, and dividing streamlines $\Psi = 0$ cut at right angles at these points, since $(\nabla_2^2\Psi)_{\chi=0} = 0$. Streamlines contained between dividing streamlines passing through two consecutive stagnation points consist of a family of closed curves enclosing a further stagnation point in the interior of the fluid.

The above analysis indicates that the time-averaged paths of the fluid particles are twisted curves lying on the cylindrical surfaces $\Psi(R,\chi) = \text{constant}$. For $R < 1$, the components of the mass transport velocity $\mathbf{Q}_l$ in cylindrical polar co-ordinates are all $O(1)$; but for $R \gg 1$, the long-shore component U_l dominates (except near the bottom), so that the particle paths in the distant field are (approximately) straight and parallel to the shoreline.

4.1. *The case $\beta \ll 1$*

Ursell (1952) showed that when $\beta < \frac{1}{6}\pi$ more than one mode of edge wave is possible. For $\beta \ll 1$, the velocity potential for the modes is obtained from the approximate result

$$\phi_1 = (-i/\beta)F_n(2R)/(2n+1), \quad F_n(\lambda) = e^{-\frac{1}{2}\lambda}L_n(\lambda)/n!,$$

provided that $\sigma^2/gk = \beta(2n+1) \ll 1$. The quantity $L_n(\lambda)$ represents the Laguerre polynomial of order $n = 0, 1, 2, \ldots$, and Stokes' edge wave corresponds to $n = 0$. The Stokes drift velocity is given by Kenyon (1969) as

$$U_s = e^{-2R}[L_n'^2+(L_n-L_n')^2]_{\lambda=2R}/[\beta(2n+1)\,n!]^2,$$

where $L_n' = dL_n/d\lambda$. Calculations of the mass transport velocity according to viscous theory may be summarized as follows.

Long-shore drift. Consistent with the assumption $\beta \ll 1$, the boundary-value problem for $\langle u_{20}\rangle$ is (approximately)

$$\partial^2\langle u_{20}\rangle/\partial X^2 = 0 \quad (0 > X > -1),$$
$$\partial\langle u_{20}\rangle/\partial X = 0 \quad (X = 0),$$
$$\langle u_{20}\rangle = -3e^{-2R}[L_n(L_n''-L_n')]_{\lambda=2R}/[\beta(2n+1)\,n!]^2 \quad (X = -1),$$

where $X = \chi/\beta$. Thus $\langle u_{20}\rangle$ is independent of χ, and the mass transport velocity in the long-shore direction is given by

$$U_l = e^{-2R}[L_n'^2 + (L_n - L_n')^2 - 3L_n(L_n'' - L_n')]_{\lambda=2R}/[\beta(2n+1)n!]^2.$$

For $n = 0$, $U_l = U_s = e^{-2R}/\beta^2$. But for $n \geqslant 1$, the long-shore drift near the shoreline is considerably less than the corresponding value predicted by inviscid theory; far from the shoreline, viscous theory gives the larger value.

Mass transport in transverse planes. The boundary-value problem for Ψ is

$$\partial^4\Psi/\partial X^4 = 0 \quad (0 > X > -1),$$
$$\Psi = 0 \quad (X = 0,\ -1),$$
$$\partial^2\Psi/\partial X^2 = 0 \quad (X = 0),$$
$$\partial\Psi/\partial X = \beta^{-1}F(R) \quad (X = -1),$$

where $$F(R) = Re^{-2R}(\tfrac{1}{2}L_n - L_n')(L_n - 6L_n' + 6L_n'')/[(2n+1)n!]^2,$$

and the solution is $$\Psi = \tfrac{1}{2}\beta^{-1}F(R)X(X^2-1).$$

According to this, the streamlines of the mean transverse flow form a cellular structure, there being $2n+1$ cells associated with the nth mode. Those stagnation points which represent limiting closed streamlines are located on $X = -1/\sqrt{3}$, and occur at $R = 0{\cdot}5$ ($n = 0$), $R \approx 0{\cdot}32$, $1{\cdot}97$, $4{\cdot}22$ ($n = 1$), ... For $n = 1$, the boundaries of the cells are $R = \frac{3}{2}$ and $\frac{7}{2}$. The sense of the circulation in the cell containing the shoreline is the same for all modes (the radial component of velocity near the bottom being in the down-slope direction).

Remarks. The method of this subsection, based on the approximation $\beta \ll 1$ and the assumption that radial rates of change of $\langle u_{20}\rangle$ and Ψ are much smaller than the corresponding azimuthal rates of change, does not accurately predict the distant field of $\mathbf{Q}_l$ for any edge-wave mode. In particular, the calculation fails to yield the algebraic decay of the long-shore drift and the infinite sequence of stagnation points on the free surface for the mean transverse flow. Although R and βR are the correct radial and azimuthal length scales for variations in $\langle u_{20}\rangle$ and Ψ at sufficiently small values of R, the approximate method fails when $R \gg 1$ because the results of this section (together with similar considerations for $n \geqslant 1$) show that βR is the appropriate scale for both radial and azimuthal variations in the distant field. Moreover, application of boundary-layer theory breaks down near the shoreline $R = 0$, where the present results have little or no practical relevance.

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